

Causality of Massive Spin 2 Field in External Gravity

I.L. BUCHBINDER^{a,b}, D.M. GITMAN^a AND V.D. PERSHIN^c ¹

^a*Instituto de Física, Universidade de São Paulo,*

P.O. Box 66318, 05315-970, São Paulo, SP, Brasil

^b*Department of Theoretical Physics, Tomsk State Pedagogical University,*
Tomsk 634041, Russia

^c*Department of Theoretical Physics, Tomsk State University,*
Tomsk 634050, Russia

Abstract

We investigate the structure of equations of motion and lagrangian constraints in a general theory of massive spin 2 field interacting with external gravity. We demonstrate how consistency with the flat limit can be achieved in a number of specific spacetimes. One such example is an arbitrary static spacetime though equations of motion in this case may lack causal properties. Another example is provided by external gravity fulfilling vacuum Einstein equations with arbitrary cosmological constant. In the latter case there exists one-parameter family of theories describing causal propagation of the correct number of degrees of freedom for the massive spin 2 field in arbitrary dimension. For a specific value of the parameter a gauge invariance with a vector parameter appears, this value is interpreted as massless limit of the theory. Another specific value of the parameter produces gauge invariance with a scalar parameter and this cannot be interpreted as a consistent massive or massless theory.

¹e-mail: joseph@tspu.edu.ru, gitman@fma.if.usp.br, pershin@ic.tsu.ru

Problems of consistent equations of motion for interacting higher spin fields deserve studying due to many reasons. First of all, string theory includes an infinite tower of massive excitations with all possible spins and thus should allow some consistent effective description of arbitrary spin fields interaction. Second, composite resonance particles with higher spins do exist and one should be able to describe their interaction (for example, with external electromagnetic and gravitational fields) in terms of some effective local field theory. At last, investigation of higher spin fields is interesting on its own from the general point of view. It would be surprising if nature admits description of free fields with arbitrary spins but stops, say, at spin 1 in case of interacting massive fields. Even if it is really the case, one should try to understand why this is the way the nature works.

This note is devoted to investigation of the massive spin 2 field interacting with external gravity which represents one of the simplest higher spin models. It has been studied in numerous papers [1]-[6]² but careful and general analysis of the consistency and causality of the theory in arbitrary curved spacetime was still absent.

There are at least two ways the interaction may spoil the consistency of a higher spin theory. Firstly, interaction may change the number of dynamical degrees of freedom. For example, a massive field with spin s in $D = 4$ Minkowski spacetime is described by a rank s symmetric traceless transverse tensor $\phi_{(\mu_1 \dots \mu_s)}$ satisfying the mass shell condition:

$$(\partial^2 - m^2)\phi_{\mu_1 \dots \mu_s} = 0, \quad \partial^\mu \phi_{\mu \mu_1 \dots \mu_{s-1}} = 0, \quad \phi^\mu_{\mu \mu_1 \dots \mu_{s-2}} = 0. \quad (1)$$

To reproduce all these equations from a single lagrangian one needs to introduce auxiliary fields $\chi_{\mu_1 \dots \mu_{s-2}}, \chi_{\mu_1 \dots \mu_{s-3}}, \dots, \chi$ [11, 12]. These symmetric traceless fields vanish on shell but their presence in the theory provides lagrangian description of the conditions (1). In higher dimensional spacetimes there appear fields of more complex tensor structure but general situation remains the same, i.e. lagrangian description always requires presence of unphysical auxiliary degrees of freedom.

Namely these auxiliary fields create problems when one tries to turn on interaction in the theory. Arbitrary interaction makes the auxiliary fields dynamical thus increasing the number of degrees of freedom. Usually these extra degrees of freedom are ghostlike and should be considered as pathological. Requirement of absence of these extra dynamical degrees of freedom imposes severe restrictions on the possible interaction.

To construct a consistent massive field theory one often starts with a corresponding consistent massless theory which should be invariant with respect to gauge symmetry and then breaks this symmetry by introducing mass terms into lagrangian. In this case invariance of the kinetic part of lagrangian guarantees the correct number of degrees of freedom in both massless and massive theories. For example, spin 2 field possesses a gauge invariant massless lagrangian only if external gravity satisfies vacuum Einstein equations and so it is usually believed that the massive spin 2 field can consistently propagate also only in Einstein spacetimes.

In this paper which is a sequel to [10] we show that this belief is not true and that in massive case there exists a number of possibilities of providing the correct number of degrees of freedom. Of course, invariance of kinetic term does always provide the correct number of degrees of freedom but this is not the only possibility. In particular, we describe below an example when external spacetime does not fulfill Einstein equations but the theory is consistent with the flat spacetime limit.

²For analysis of the corresponding massless models see e.g. [7, 8, 9]

The other problem that may arise in higher spin fields theories is connected with possible violation of causal properties. This problem was first noted in the theory of spin 3/2 field in external fields [13] (see also the review [14] and a recent discussion in [15])

In general, when one has a system of differential equations for a set of fields ϕ^B (to be specific, let us say about second order equations)

$$M_{AB}{}^{\mu\nu}\partial_\mu\partial_\nu\phi^B + \dots = 0, \quad \mu, \nu = 0, \dots, D-1 \quad (2)$$

the following definitions are used. A characteristic matrix is the matrix function of D arguments n_μ built out of the coefficients at the second derivatives in the equations: $M_{AB}(n) = M_{AB}{}^{\mu\nu}n_\mu n_\nu$. A characteristic equation is $\det M_{AB}(n) = 0$. A characteristic surface is the surface $S(x) = \text{const}$ where $\partial_\mu S(x) = n_\mu$.

If for any n_i ($i = 1, \dots, D-1$) all solutions of the characteristic equation $n_0(n_i)$ are real then the system of differential equations is called hyperbolic and describes propagation of some wave processes. The hyperbolic system is called causal if there is no timelike vectors among solutions n_μ of the characteristic equations. Such a system describes propagation with a velocity not exceeding the speed of light. If there exist timelike solutions for n_μ then the corresponding characteristic surfaces are spacelike and violate causality.

Turning on interaction in theories of higher spin fields in general changes the characteristic matrix and there appears possibility of superluminal propagation. Such a situation also should be considered as pathological.

Both these problems arise in the theory of massive spin 2 field coupled to external gravitational field. To provide consistency of the interaction we should conserve the same number of physical degrees of freedom and constraints that the theory possesses in flat spacetime. To find the complete set of constraints we will use the general lagrangian scheme [16] which is equivalent to the Dirac-Bergmann procedure in hamiltonian formalism but for our purposes is simpler. In the case of second class constraints (which is relevant for massive higher spin fields) it consists in the following steps. If in a theory of some set of fields $\phi^A(x)$, $A = 1, \dots, N$ the original lagrangian equations of motion define only $r < N$ of the second time derivatives ("accelerations") $\ddot{\phi}^A$ then one can build $N - r$ primary constraints, i.e. linear combinations of the equations of motion that does not contain accelerations. Requirement of conservation in time of the primary constraints either define some of the missing accelerations or lead to new (secondary) constraints. Then one demands conservation of the secondary constraints and so on, until all the accelerations are defined and the procedure closes up.

Before considering the theory in external gravitational field we analyze the structure of equations of motion in Minkowski spacetime. The purpose of this analysis is twofold. First, we illustrate the general scheme of calculating the constraints within covariant lagrangian framework. In addition, building a consistent and causal theory in curved spacetime we use these flat constraints as a reference point.

Free spin 2 field is known to be described by the Fierz-Pauli action [11] (we consider arbitrary spacetime dimension):

$$S = \int d^D x \left\{ \frac{1}{4} \partial_\mu H \partial^\mu H - \frac{1}{4} \partial_\mu H_{\nu\rho} \partial^\mu H^{\nu\rho} - \frac{1}{2} \partial^\mu H_{\mu\nu} \partial^\nu H + \frac{1}{2} \partial_\mu H_{\nu\rho} \partial^\rho H^{\nu\mu} - \frac{m^2}{4} H_{\mu\nu} H^{\mu\nu} + \frac{m^2}{4} H^2 \right\}. \quad (3)$$

Here the role of auxiliary field is played by the trace $H = \eta^{\mu\nu} H_{\mu\nu}$. The equations of motion

$$E_{\mu\nu} = \partial^2 H_{\mu\nu} - \eta_{\mu\nu} \partial^2 H + \partial_\mu \partial_\nu H + \eta_{\mu\nu} \partial^\alpha \partial^\beta H_{\alpha\beta} - \partial_\sigma \partial_\mu H^\sigma{}_\nu - \partial_\sigma \partial_\nu H^\sigma{}_\mu - m^2 H_{\mu\nu} + m^2 H \eta_{\mu\nu} = 0 \quad (4)$$

contain D primary constraints (expressions without second time derivatives $\ddot{H}_{\mu\nu}$):

$$E_{00} = \Delta H_{ii} - \partial_i \partial_j H_{ij} - m^2 H_{ii} \equiv \varphi_0^{(1)} \approx 0 \quad (5)$$

$$E_{0i} = \Delta H_{0i} + \partial_i \dot{H}_{kk} - \partial_k \dot{H}_{ki} - \partial_i \partial_k H_{0k} - m^2 H_{0i} \equiv \varphi_i^{(1)} \approx 0. \quad (6)$$

The remaining equations of motion $E_{ij} = 0$ allow to define the accelerations \ddot{H}_{ij} in terms of $\dot{H}_{\mu\nu}$ and $H_{\mu\nu}$. The accelerations \ddot{H}_{00} , \ddot{H}_{0i} cannot be expressed from the equations directly.

Conditions of conservation of the primary constraints in time $\dot{E}_{0\mu} \approx 0$ lead to D secondary constraints. On-shell they are equivalent to

$$\varphi_\nu^{(2)} = \partial^\mu E_{\mu\nu} = m^2 \partial_\nu H - m^2 \partial^\mu H_{\mu\nu} \approx 0 \quad (7)$$

Conservation of $\varphi_i^{(2)}$ defines $D - 1$ accelerations \ddot{H}_{0i} and conservation of $\varphi_0^{(2)}$ gives another one constraint. It is convenient to choose it in the covariant form by adding suitable terms proportional to the equations of motion:

$$\varphi^{(3)} = \partial^\mu \partial^\nu E_{\mu\nu} + \frac{m^2}{D-2} \eta^{\mu\nu} E_{\mu\nu} = H m^4 \frac{D-1}{D-2} \approx 0 \quad (8)$$

Conservation of $\varphi^{(3)}$ gives one more constraint on initial values

$$\varphi^{(4)} = -\dot{H}_{00} + \dot{H}_{kk} = \dot{H} \approx 0 \quad (9)$$

and from the conservation of this last constraint the acceleration \ddot{H}_{00} is defined.

Altogether there are $2D + 2$ constraints on the initial values of $\dot{H}_{\mu\nu}$ and $H_{\mu\nu}$. The lagrangian theory is equivalent to the system of the equations

$$(\partial^2 - m^2) H_{\mu\nu} = 0, \quad \partial^\mu H_{\mu\nu} = 0, \quad H^\mu{}_\mu = 0. \quad (10)$$

and describes traceless and transverse symmetric tensor field of the second rank.

Obviously, the equations of motion (10) are causal because the characteristic equation

$$\det M(n) = (n^2)^{D(D+1)/2} \quad (11)$$

has 2 multiply degenerate roots

$$-n_0^2 + n_i^2 = 0, \quad n_0 = \pm \sqrt{n_i^2}. \quad (12)$$

which correspond to real null solutions for n_μ . Note that analysis of causality is possible only after calculation of all the constraints. Original lagrangian equations of motion (4) have degenerate characteristic matrix $\det M(n) \equiv 0$ and do not allow to define propagation cones of the field $H_{\mu\nu}$.

In the massless limit $m^2 = 0$ the structure of the theory (3) changes. Instead of the secondary constraints (7) conservation of the primary constraints lead to identities $\partial^\mu E_{\mu\nu} \equiv 0$ which mean that the theory becomes gauge invariant with respect to the local transformations $\delta H_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$. Such a theory represents the quadratic part of the Einstein-Hilbert action for gravitational field and the gauge invariance is a linear counterpart of the general coordinate invariance.

Now if we want to construct a theory of massive spin 2 field on a curved manifold first of all we should provide the same number of propagating degrees of freedom as in the flat case. It means that new equations of motion $E_{\mu\nu}$ should lead to exactly $2D + 2$ constraints and in the flat spacetime limit these constraints should reduce to their flat counterparts.

Generalizing (3) to curved spacetime we should substitute all derivatives by the co-variant ones and also we can add non-minimal terms containing curvature tensor with some dimensionless coefficients in front of them. As a result, the most general action for massive spin 2 field in curved spacetime quadratic in derivatives and consistent with the flat limit should have the form [1]:

$$S = \int d^D x \sqrt{-G} \left\{ \frac{1}{4} \nabla_\mu H \nabla^\mu H - \frac{1}{4} \nabla_\mu H_{\nu\rho} \nabla^\mu H^{\nu\rho} - \frac{1}{2} \nabla^\mu H_{\mu\nu} \nabla^\nu H + \frac{1}{2} \nabla_\mu H_{\nu\rho} \nabla^\rho H^{\nu\mu} \right. \\ \left. + \frac{a_1}{2} R H_{\alpha\beta} H^{\alpha\beta} + \frac{a_2}{2} R H^2 + \frac{a_3}{2} R^{\mu\alpha\nu\beta} H_{\mu\nu} H_{\alpha\beta} + \frac{a_4}{2} R^{\alpha\beta} H_{\alpha\sigma} H_\beta^\sigma + \frac{a_5}{2} R^{\alpha\beta} H_{\alpha\beta} H \right. \\ \left. - \frac{m^2}{4} H_{\mu\nu} H^{\mu\nu} + \frac{m^2}{4} H^2 \right\} \quad (13)$$

where a_1, \dots, a_5 are so far arbitrary dimensionless coefficients, $R^\mu{}_{\nu\lambda\kappa} = \partial_\lambda \Gamma^\mu_{\nu\kappa} - \dots$, $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$.

Equations of motion

$$E_{\mu\nu} = \nabla^2 H_{\mu\nu} - G_{\mu\nu} \nabla^2 H + \nabla_\mu \nabla_\nu H + G_{\mu\nu} \nabla^\alpha \nabla^\beta H_{\alpha\beta} - \nabla_\sigma \nabla_\mu H^\sigma{}_\nu - \nabla_\sigma \nabla_\nu H^\sigma{}_\mu \\ + 2a_1 R H_{\mu\nu} + 2a_2 G_{\mu\nu} R H + 2a_3 R_\mu{}^\alpha{}_\nu{}^\beta H_{\alpha\beta} + a_4 R_\mu{}^\alpha H_{\alpha\nu} + a_4 R_\nu{}^\alpha H_{\alpha\mu} \\ + a_5 R_{\mu\nu} H + a_5 G_{\mu\nu} R^{\alpha\beta} H_{\alpha\beta} - m^2 H_{\mu\nu} + m^2 H G_{\mu\nu} \approx 0 \quad (14)$$

contain second time derivatives of $H_{\mu\nu}$ in the following way:

$$E_{00} = (G^{mn} - G_{00} G^{00} G^{mn} + G_{00} G^{0m} G^{0n}) \nabla_0 \nabla_0 H_{mn} + O(\nabla_0), \\ E_{0i} = (-G_{0i} G^{00} G^{mn} + G_{0i} G^{0m} G^{0n} - G^{0m} \delta_i^n) \nabla_0 \nabla_0 H_{mn} + O(\nabla_0), \\ E_{ij} = (G^{00} \delta_i^m \delta_j^n - G_{ij} G^{00} G^{mn} + G_{ij} G^{0m} G^{0n}) \nabla_0 \nabla_0 H_{mn} + O(\nabla_0). \quad (15)$$

So we see that accelerations \ddot{H}_{00} and \ddot{H}_{0i} again (as in the flat case) do not enter the equations of motion while accelerations \ddot{H}_{ij} can be expressed through $\dot{H}_{\mu\nu}$, $H_{\mu\nu}$ and their spatial derivatives.

There are D linear combinations of the equations of motion which do not contain second time derivatives and so represent primary constraints of the theory:

$$\varphi_\mu^{(1)} = E^0{}_\mu = G^{00} E_{0\mu} + G^{0j} E_{j\mu} \quad (16)$$

Now one should calculate time derivatives of these constraints and define secondary ones. In order to do this in a covariant form we can add to the time derivative of $\varphi_\mu^{(1)}$ any linear

combination of equations of motion and primary constraints. So we choose the secondary constraints in the following way:

$$\begin{aligned}
\varphi_\mu^{(2)} &= \nabla^\alpha E_{\alpha\mu} = \dot{\varphi}_\mu^{(1)} + \partial_i E^i{}_\mu + \Gamma_{\alpha 0}^\alpha \varphi_\mu^{(1)} + \Gamma_{\alpha i}^\alpha E^i{}_\mu - \Gamma_{\mu 0}^\sigma \varphi_\sigma^{(1)} - \Gamma_{\mu i}^\sigma E^i{}_\sigma \\
&= (2a_1 R - m^2) \nabla^\mu H_{\mu\nu} + (2a_2 R + m^2) \nabla_\nu H + 2a_3 R^{\mu\alpha}{}_\nu{}^\beta \nabla_\mu H_{\alpha\beta} + a_4 R^{\mu\alpha} \nabla_\mu H_{\alpha\nu} \\
&\quad + (a_4 - 2) R^\alpha{}_\nu \nabla^\mu H_{\alpha\mu} + a_5 R^{\alpha\mu} \nabla_\nu H_{\alpha\mu} + (a_5 + 1) R^\alpha{}_\nu \nabla_\alpha H \\
&\quad + (2a_1 + \frac{a_4}{2}) H_{\alpha\nu} \nabla^\alpha R + (2a_2 + \frac{a_5}{2}) H \nabla_\nu R \\
&\quad + H_{\alpha\beta} \left[(2a_3 + a_5 + 1) \nabla_\nu R^{\alpha\beta} + (a_4 - 2a_3 - 2) \nabla^\alpha R^\beta{}_\nu \right]
\end{aligned} \tag{17}$$

At the next step conservation of these D secondary constraints should lead to one new constraint and to expressions for $D-1$ accelerations \ddot{H}_{0i} . This means that the constraints (17) should contain the first time derivatives $\dot{H}_{0\mu}$ through the matrix with the rank $D-1$:

$$\begin{aligned}
\varphi_0^{(2)} &= A \dot{H}_{00} + B^j \dot{H}_{0j} + \dots \\
\varphi_i^{(2)} &= C_i \dot{H}_{00} + D_i^j \dot{H}_{0j} + \dots
\end{aligned} \tag{18}$$

$$\text{rank } \hat{\Phi}_\mu{}^\nu \equiv \text{rank} \begin{vmatrix} A & B^j \\ C_i & D_i^j \end{vmatrix} = D - 1 \tag{19}$$

In the flat spacetime we had the matrix

$$\hat{\Phi}_\mu{}^\nu = \begin{vmatrix} 0 & 0 \\ 0 & m^2 \delta_i^j \end{vmatrix} \tag{20}$$

In the curved case the explicit form of this matrix elements in the constraints (17) is:

$$\begin{aligned}
A &= R G^{00} (2a_1 + 2a_2) + R^{00} (a_4 + a_5) + R^0{}_0 G^{00} (a_4 + a_5 - 1) \\
B^j &= m^2 G^{0j} + R G^{0j} (2a_1 + 4a_2) + 2a_3 R^{0j}{}_0{}^0 + R^j{}_0 G^{00} (a_4 - 2) \\
&\quad + R^{0j} (a_4 + 2a_5) + R^0{}_0 G^{0j} (a_4 + 2a_5) \\
C_i &= R^0{}_i G^{00} (a_4 + a_5 - 1) \\
D_i^j &= -m^2 G^{00} \delta_i^j + 2a_1 R G^{00} \delta_i^j + 2a_3 R^{0j}{}_i{}^0 + a_4 R^{00} \delta_i^j \\
&\quad + (a_4 - 2) R^j{}_i G^{00} + (a_4 + 2a_5) R_i^0 G^{0j}
\end{aligned} \tag{21}$$

At this stage the restrictions that consistency imposes on the type of interaction reduce to the requirements that the above matrix elements give

$$\det \hat{\Phi} = 0, \quad \det D_i^j \neq 0 \tag{22}$$

When the gravitational background is arbitrary it is not clear how to fulfill this condition by choosing some specific values of non-minimal couplings a_1, \dots, a_5 . For example, requirement of vanishing of the elements A and C_i (21) would lead to contradictory equations $a_4 + a_5 = 0$, $a_4 + a_5 - 1 = 0$.

But the consistency conditions (22) can be fulfilled in a number of specific gravitational background. Namely, any spacetime which in some coordinates has

$$R^0{}_i = 0 \tag{23}$$

provides such an example. In such a spacetime $R^{00} = R^0_0 G^{00}$ and choosing coefficients $a_1 + a_2 = 0$, $2a_4 + 2a_5 = 1$ we have the first column of the matrix $\hat{\Phi}$ vanishing and so the conditions (22) fulfilled.

As a first example where (23) holds let us consider an arbitrary static spacetime, i.e. a spacetime having a timelike Killing vector and invariant with respect to the time reversal $x^0 \rightarrow -x^0$. In such a spacetime one can always find coordinates where

$$\partial_0 G_{\mu\nu} = 0, \quad G_{0i} = 0. \quad (24)$$

The matrix elements (21) in this case become

$$\begin{aligned} A &= R G^{00}(2a_1 + 2a_2) + R^{00}(2a_4 + 2a_5 - 1), \quad B^j = 0, \quad C_i = 0, \\ D_i^j &= (-m^2 G^{00} + 2a_1 R G^{00} + a_4 R^{00})\delta_i^j + (a_4 - 2)R^j_i G^{00} + 2a_3 R^{0j}_i \end{aligned} \quad (25)$$

and (22) lead to the following conditions:

$$2a_1 + 2a_2 = 0, \quad 2a_4 + 2a_5 - 1 = 0, \quad \det D_i^j \neq 0 \quad (26)$$

The last inequality may be violated in strong gravitational field and as we comment below this fact may lead to causal problems.

Suppose that all the conditions (26) are fulfilled. For simplicity we also choose $a_3 = 0$. Then we have the equations of the form (14) with the coefficients

$$a_1 = \frac{\xi_1}{2}, \quad a_2 = -\frac{\xi_1}{2}, \quad a_3 = 0, \quad a_4 = \frac{1}{2} - \xi_2, \quad a_5 = \xi_2 \quad (27)$$

where ξ_1, ξ_2 are two arbitrary coupling parameters.

One of the secondary constraints

$$\begin{aligned} \varphi_0^{(2)} &= \nabla^\alpha E_{\alpha 0} = \nabla_0 H_{ij} \left[G^{ij}(m^2 - \xi_1 R) + (1 + \xi_2) G^{ij} R^0_0 + \xi_2 R^{ij} \right] \\ &\quad + \nabla_i H_{0j} \left[G^{ij}(\xi_1 R - m^2) - \left(\frac{3}{2} + \xi_2 \right) G^{ij} R^0_0 + \left(\frac{1}{2} - \xi_2 \right) R^{ij} \right] \\ &\quad + \left(\frac{1}{4} + \xi_1 - \frac{\xi_2}{2} \right) H_{0i} \nabla^i R - \left(\frac{3}{2} + \xi_2 \right) H_{0i} \nabla^i R^0_0 \end{aligned} \quad (28)$$

does not contain velocities $\dot{H}_{00}, \dot{H}_{0i}$ and so its conservation leads to a new constraint $\varphi^{(3)} \approx \nabla_0 \nabla^\alpha E_{\alpha 0}$. After exclusion from this expression the accelerations \ddot{H}_{ij} we get this constraint as the following combination of the equations of motion:

$$\begin{aligned} \varphi^{(3)} &= \nabla_0 \nabla^\mu E_{\mu 0} - \xi_2 G_{00} R^{ij} E_{ij} + \frac{1}{D-2} \left[m^2 G_{00} + (\xi_2 - \xi_1) R G_{00} + R_{00} \right] G^{ij} E_{ij} \\ &= \left\{ \frac{D-1}{D-2} m^4 + \frac{2\xi_2 - 2\xi_1(D-1)}{D-2} m^2 R + \left(2\xi_2 + \frac{D-1}{D-2} \right) m^2 R^0_0 - \xi_2^2 R_{ij} R^{ij} \right. \\ &\quad \left. + \xi_1^2 R R + \left(\frac{\xi_2 - \xi_1(D-1)}{D-2} - 2\xi_1 \xi_2 \right) R R^0_0 + \xi_2(\xi_2 + 1) R_{00} R^{00} \right\} H_{00} + \dots \end{aligned} \quad (29)$$

We did not write down the explicit form of this constraint because everything we should know about it is the way it contains the component H_{00} . Namely, $\varphi^{(3)}$ contains

neither the acceleration \ddot{H}_{00} nor the velocity \dot{H}_{00} . It means that its conservation in time leads to another new constraints

$$\varphi^{(4)} \approx \nabla_0 \varphi^{(3)} \quad (30)$$

and hence the total number of constraints is the same as in the flat spacetime provided that the expression in the braces in front of H_{00} in $\varphi^{(3)}$ does not vanish.

Vanishing of this expression in braces as well as violation of the inequality (26) leads to local changing of the number of degrees of freedom and this fact is known to be related with acausal behavior in higher spin theories in external fields [15, 17]. In general, causality breaks in those cases when there are points in spacetime in which it is impossible to define all the accelerations from the conservation of constraints.

In our case it means that causality will hold everywhere only if $\det D^i_j \neq 0$ and the expression in braces in $\varphi^{(3)}$ also does not vanish. Obviously, in general case there are values of $R_{\mu\nu}$ that violate these requirements.

It is instructive to consider in more detail the Reissner-Nordstrom solution in $D = 4$ as a simple example of non-trivial static spacetime

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} + r^2 d\Omega^2 \quad (31)$$

In this case causality problems are absent when the expressions

$$\begin{aligned} \det D_i^j &\sim \left(m^2 - (1 + 2\xi_2)\frac{Q^2}{r^4}\right)\left(m^2 + 2\frac{Q^2}{r^4}\right)^2 \\ \varphi^{(3)} &= \left\{\frac{D-1}{D-2}m^4 + m^2\left(2\xi_2 + \frac{D-1}{D-2}\right)\frac{Q^2}{r^4} + \xi_2(1 - 2\xi_2)\frac{Q^4}{r^8}\right\}H_{00} + \dots \end{aligned} \quad (32)$$

do not vanish. Far enough from the horizon where all the terms containing r in (32) are too small and so propagation is causal in this region. Causal problems may develop only for small values of r . This might be excluded if all terms in (32) were positive, that is if

$$1 + 2\xi_2 < 0, \quad 2\xi_2 + \frac{D-1}{D-2} > 0, \quad \xi_2(1 - 2\xi_2) > 0 \quad (33)$$

but these three conditions are contradictory.

It means that for any value of the coupling parameter expressions in (32) vanish for some values of the coordinate r and the massive spin 2 field propagate causally only in the regions near infinity but close to the horizon causality is lost. Of course, this example does not mean that there cannot exist other spacetimes where causality might be achieved everywhere for some special values of coupling parameters.

Another possible way to fulfill the consistency requirements (22) is to consider spacetimes representing solutions of vacuum Einstein equations with arbitrary cosmological constant:

$$R_{\mu\nu} = \frac{1}{D}G_{\mu\nu}R. \quad (34)$$

In this case the coefficients a_4, a_5 in the lagrangian (13) are absent and the matrix $\hat{\Phi}$

takes the form:

$$\hat{\Phi}_\mu{}^\nu = \left\| \begin{array}{c|c} RG^{00}(2a_1 + 2a_2 - \frac{1}{D}) & RG^{0j}(2a_1 + 4a_2) + 2a_3 R^{0j}{}_0 + m^2 G^{0j} \\ \hline 0 & 2a_3 R^{0j}{}_i + RG^{00}\delta_i^j(2a_1 - \frac{2}{D}) - m^2 G^{00}\delta_i^j \end{array} \right\| \quad (35)$$

The simplest way to make the rank of this matrix to be equal to $D - 1$ is provided by the following choice of the coefficients:

$$2a_1 + 2a_2 - \frac{1}{D} = 0, \quad a_3 = 0, \quad 2R\left(a_1 - \frac{1}{D}\right) - m^2 \neq 0. \quad (36)$$

As a result, we have one-parameter family of theories:

$$\begin{aligned} a_1 &= \frac{\xi}{D}, \quad a_2 = \frac{1 - 2\xi}{2D}, \quad a_3 = 0, \quad a_4 = 0, \quad a_5 = 0 \\ R_{\mu\nu} &= \frac{1}{D}G_{\mu\nu}R, \quad \frac{2(1 - \xi)}{D}R + m^2 \neq 0. \end{aligned} \quad (37)$$

with ξ an arbitrary real number.

The action in this case takes the form

$$\begin{aligned} S = \int d^D x \sqrt{-G} \Big\{ & \frac{1}{4} \nabla_\mu H \nabla^\mu H - \frac{1}{4} \nabla_\mu H_{\nu\rho} \nabla^\mu H^{\nu\rho} - \frac{1}{2} \nabla^\mu H_{\mu\nu} \nabla^\nu H + \frac{1}{2} \nabla_\mu H_{\nu\rho} \nabla^\rho H^{\nu\mu} \\ & + \frac{\xi}{2D} R H_{\mu\nu} H^{\mu\nu} + \frac{1 - 2\xi}{4D} R H^2 - \frac{m^2}{4} H_{\mu\nu} H^{\mu\nu} + \frac{m^2}{4} H^2 \Big\}. \end{aligned} \quad (38)$$

and the corresponding equations of motion are

$$\begin{aligned} E_{\mu\nu} &= \nabla^2 H_{\mu\nu} - G_{\mu\nu} \nabla^2 H + \nabla_\mu \nabla_\nu H + G_{\mu\nu} \nabla^\alpha \nabla^\beta H_{\alpha\beta} - \nabla_\sigma \nabla_\mu H^\sigma{}_\nu - \nabla_\sigma \nabla_\nu H^\sigma{}_\mu \\ &+ \frac{2\xi}{D} R H_{\mu\nu} + \frac{1 - 2\xi}{D} R H G_{\mu\nu} - m^2 H_{\mu\nu} + m^2 H G_{\mu\nu} = 0 \end{aligned} \quad (39)$$

The secondary constraints built out of them are

$$\varphi_\mu^{(2)} = \nabla^\alpha E_{\alpha\mu} = (\nabla_\mu H - \nabla^\alpha H_{\mu\alpha}) \left(m^2 + \frac{2(1 - \xi)}{D} R \right) \quad (40)$$

and the matrix $\hat{\Phi}$ looks like

$$\hat{\Phi}_\mu{}^\nu = \left(m^2 + \frac{2(1 - \xi)}{D} R \right) \left\| \begin{array}{c|c} 0 & G^{0j} \\ \hline 0 & -G^{00}\delta_i^j \end{array} \right\| \quad (41)$$

Just like in the flat case, in this theory the conditions $\dot{\varphi}_i^{(2)} \approx 0$ define the accelerations \ddot{H}_{0i} and the condition $\dot{\varphi}_0^{(2)} \approx 0$ after excluding \ddot{H}_{0i} gives a new constraint, i.e. the acceleration \ddot{H}_{00} is not defined at this stage.

To define the new constraint in a covariant form we use the following linear combination of $\dot{\varphi}_\mu^{(2)}$, equations of motion, primary and secondary constraints:

$$\begin{aligned} \varphi^{(3)} &= \frac{m^2}{D - 2} G^{\mu\nu} E_{\mu\nu} + \nabla^\mu \nabla^\nu E_{\mu\nu} + \frac{2(1 - \xi)}{D(D - 2)} R G^{\mu\nu} E_{\mu\nu} = \\ &= H \frac{1}{D - 2} \left(\frac{2(1 - \xi)}{D} R + m^2 \right) \left(\frac{D + 2\xi(1 - D)}{D} R + m^2(D - 1) \right) \approx 0. \end{aligned} \quad (42)$$

This gives tracelessness condition for the field $H_{\mu\nu}$ provided that parameters of the theory fulfill the conditions:

$$\frac{2(1-\xi)}{D}R + m^2 \neq 0, \quad \frac{D+2\xi(1-D)}{D}R + m^2(D-1) \neq 0 \quad (43)$$

Requirement of conservation of $\varphi^{(3)}$ leads to one more constraint

$$\dot{\varphi}^{(3)} \sim \dot{H} \implies \varphi^{(4)} = \dot{H} \approx 0. \quad (44)$$

The last acceleration \ddot{H}_{00} is expressed from the condition $\dot{\varphi}^{(4)} \approx 0$.

Using the constraints for simplifying the equations of motion we see that the original equations are equivalent to the following system:

$$\begin{aligned} \nabla^2 H_{\mu\nu} + 2R^\alpha{}_\mu{}^\beta{}_\nu H_{\alpha\beta} + \frac{2(\xi-1)}{D}RH_{\mu\nu} - m^2 H_{\mu\nu} &= 0, \\ H^\mu{}_\mu &= 0, \quad \dot{H}^\mu{}_\mu = 0, \quad \nabla^\mu H_{\mu\nu} = 0, \\ G^{00}\nabla_0\nabla_i H^i{}_\nu - G^{0i}\nabla_0\nabla_i H^0{}_\nu - G^{0i}\nabla_i\nabla_0 H^0{}_\nu - G^{ij}\nabla_i\nabla_j H^0{}_\nu - 2R^{\alpha 0\beta}{}_\nu H_{\alpha\beta} \\ &\quad - \frac{2(\xi-1)}{D}RH^0{}_\nu + m^2 H^0{}_\nu = 0. \end{aligned} \quad (45)$$

The last expression represents D primary constraints.

For any values of ξ (except two degenerate values excluded by (43)) the theory describes the same number of degrees of freedom as in the flat case - the symmetric, covariantly transverse and traceless tensor. D primary constraints guarantees conservation of the transversality conditions in time.

Let us now consider the causal properties of the theory. Again, if we tried to use the equations of motion in the original lagrangian form (39) then the characteristic matrix

$$M_{\mu\nu}{}^{\lambda\kappa}(n) = \delta_{(\mu\nu)}^{(\lambda\kappa)}n^2 - G_{\mu\nu}G^{\lambda\kappa}n^2 + G^{\lambda\kappa}n_\mu n_\nu + G_{\mu\nu}n^\lambda n^\kappa - \delta_\nu^{(\kappa}n^{\lambda)}n_\mu - \delta_\mu^{(\kappa}n^{\lambda)}n_\nu \quad (46)$$

would be degenerate. This fact can be seen from the relation

$$n^\mu M_{\mu\nu}{}^{\lambda\kappa}(n) \equiv 0 \quad (47)$$

which means that any symmetric tensor of the form $n_{(\mu}t_{\nu)}$ (with t_ν an arbitrary vector) represents a “null vector” for the matrix $M(n)$ and therefore $\det M = 0$.

After having used the constraints we obtain the equations of motion written in the form (45) and the characteristic matrix becomes non-degenerate:

$$M_{\mu\nu}{}^{\lambda\kappa}(n) = \delta_{\mu\nu}{}^{\lambda\kappa}n^2, \quad n^2 = G^{\alpha\beta}n_\alpha n_\beta. \quad (48)$$

The characteristic cones remains the same as in the flat case. At any point x_0 we can choose locally $G^{\alpha\beta}(x_0) = \eta^{\alpha\beta}$ and then

$$n^2|_{x_0} = -n_0^2 + n_i^2 \quad (49)$$

Just like in the flat case the equations are hyperbolic and causal.

Now let us discuss the massless limit of the theory under consideration. There are several points of view on the definition of masslessness in a curved spacetime of an arbitrary

dimension. We guess that the most physically accepted definition is the one referring to appearance of a gauge invariance for some specific values of the theory parameters (see e.g. [19, 20] for a recent discussion).

In our case it means that the real mass parameter M for the field $H_{\mu\nu}$ in an Einstein spacetime is defined as

$$M^2 = m^2 + \frac{2(1-\xi)}{D}R \quad (50)$$

When $M^2 = 0$ instead of D secondary constraints $\varphi_\mu^{(2)}$ we have D identities for the equations of motion $\nabla^\mu E_{\mu\nu} \equiv 0$ and the theory acquires gauge invariance $\delta H_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$. This explains the meaning of the first condition in (43), it just tells us that the theory is massive.

In fact, two parameters m^2 and ξ enter the action (38) in a single combination M^2 (50). Since scalar curvature is constant in Einstein spacetime there is no way to distinguish between the corresponding terms $\sim \xi RHH$, $\sim m^2 HH$ (with arbitrary ξ , m) in the action. The difference between the two will appear only if we consider Weyl rescaling of the metric. Note that the “massless” theory with $M^2 = 0$ is not Weyl invariant. In the case of dS/AdS spacetimes the difference between masslessness, conformal and gauge invariance and null cone propagation was discussed in detail in [21]. In our case the theory obviously cannot possess Weyl invariance.

The second inequality (43) is more mysterious. If it fails to hold, i.e. if $M^2 = M_c^2 \equiv \frac{D-2}{D(D-1)}R$ then instead of the constraint $\varphi^{(3)}$ the scalar identity

$$\nabla^\mu \nabla^\nu E_{\mu\nu} + \frac{R}{D(D-1)}G^{\mu\nu}E_{\mu\nu} = 0 \quad (51)$$

with the corresponding gauge invariance

$$\delta H_{\mu\nu} = \nabla_\mu \nabla_\nu \epsilon + \frac{R}{D(D-1)}G_{\mu\nu}\epsilon \quad (52)$$

arise.

Appearance of this gauge invariance with a scalar parameter was first found for the massive spin 2 in spacetime of constant curvature in [21] and was further investigated [2, 4] in spacetimes with positive cosmological constant. Our analysis shows that this gauge invariance is a feature of more general spin 2 theories in arbitrary Einstein spacetimes. In this case we can simplify the equations of motion using the secondary constraints (40):

$$\nabla^2 H_{\mu\nu} - \nabla_\mu \nabla_\nu H + 2R_\mu{}^\alpha{}_\nu{}^\beta H_{\alpha\beta} + \frac{2-D}{D(D-1)}RH_{\mu\nu} - \frac{1}{D(D-1)}RG_{\mu\nu}H = 0. \quad (53)$$

After imposing the gauge condition³ $H = 0$ one can see that these equations describe causal propagation of the field $H_{\mu\nu}$ but the number of propagating degrees of freedom corresponds to neither massive nor massless spin 2 free field. It was argued in [2, 4] that appearance of the gauge invariance (52) leads to such pathological properties as violation of the classical Hamiltonian positiveness and negative norm states in the quantum version

³It does not fix (52) completely and the residual symmetry with the parameter obeying $(\nabla^2 + \frac{R}{D-1})\epsilon = 0$ remains.

of the theory. One should expect similar problems in the general spin 2 theory in arbitrary Einstein spacetime described in this paper.

We demonstrated that correct number of degrees of freedom in the massive spin 2 theory (algebraic consistency) can be achieved in a large class of curved spacetimes which include as particular cases arbitrary static spacetimes and vacuum Einstein spacetimes. An analysis of the constraints structure shows that in case of a static spacetime there exists a potential source of acausal behavior. However, as we see on the example of Reissner-Nordstrom spacetime causal propagation is possible in the regions where gravitational field is weak enough. In Einstein spacetimes spin 2 massive field can be consistently described by a one-parameter family of theories (38). For any value of the parameter satisfying (43) the corresponding equations describe the correct number of degrees of freedom which propagate causally.

It is interesting to compare our approach with the paper [22] which is devoted to investigation of consistency of higher rank spin-tensor fields in curved spacetime from a different point of view. The authors of [22] considered equations for the fields carrying irreducible representations of Euclidian version of the four dimensional Lorentz group $SO(4)$ and analyzed when irreducibility of these representations is preserved in curved space. In particular, they showed that symmetric second rank tensor equations are consistent in this sense in Einstein spaces. However, such an analysis (sufficient for the proof of index theorems in [22]) is not enough when one tries to build a consistent theory for a physical field on a curved manifold starting from an irreducible representation of the Poincare group in flat spacetime. Preservation of the correct number of degrees of freedom in such a theory is a requirement independent from the algebraic consistency considered in [22]. Einstein spacetimes provide an example when both these conditions are fulfilled but as we saw in our analysis correct number of degrees of freedom can be preserved in much wider class of spacetimes. Besides, in physical theories for interacting higher spin fields we face a new problem of causality which should also be studied independently. In general there can be theories with correct number of degrees of freedom but acausal, and we really see such examples in case of spin 2 field in external gravity. It is worth to note that the authors of [22] emphasized that their analysis has no direct relation to the problem of consistent propagation of higher spin physical fields and that they did not set this problem at all.

In case of Einstein spacetimes our lagrangian for the spin 2 field in curved spacetime is the most general known so far, in all previous works only the theories with specific values of the parameter ξ were considered [4, 5]. Two degenerate values of the parameter ξ describe the theories with different degrees of freedom. One of this degenerate values corresponds to massless spin 2 field in an Einstein spacetime, another one describes neither massive nor massless spin 2 field.

The next natural step would consist in building a theory describing dynamics of both gravity and massive spin 2 field. In such a theory in addition to dynamical equations for the massive spin 2 field one would have dynamical equations for gravity with the energy-momentum tensor constructed out of spin 2 field components. The analysis of consistency then changes and one needs to have correct number of constraints and causality for both fields interacting with each other [1].

The only known consistent system of a higher spin field interacting with dynamical gravity is the theory of massless helicity 3/2 field, i.e. supergravity [23] (see also the book

[24]). In that case consistency with dynamical gravity requires four-fermion interaction. If a consistent description of spin 2 field interacting with dynamical gravity exists it may also require some non-trivial modification of the lagrangian. At least, it is known that lagrangians quadratic in spin 2 field do not provide such a consistency [1]. A possible way of consistent description of the spin 2 field on arbitrary gravitational background was recently proposed in [10]. This was achieved by means of representation of the lagrangian in the form of infinite series in curvature and imposing the consistency condition perturbatively in each order (earlier similar construction was investigated for symmetric Einstein spacetime in [6]).

Further generalizations of our analysis may include theories of massive spins $s \geq 3$ fields (which would require more complex structure of auxiliary fields) and interaction with other background fields, e.g. with scalar dilaton and antisymmetric tensor that are relevant in string theory.

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